THE COUPLING RELATION BETWEEN THE EXCESS TEMPERATURES IN AN ACTIVE ELEMENT

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The necessary and sufficient conditions to satisfy exactly the coupling relation between excess temperatures in an active element of finite dimensions under asymmetric cooling conditions are derived.

In [1], a relation between excess temperatures in a fuel element was proposed which has the form

$$\Theta(X, Y) = \frac{\Phi(X^*, Y)\Phi(X, Y^*)}{\Phi(X^*, Y^*)}, \tag{1}$$

where $\Phi(X, Y)$ is the dimensionless (excess) temperature and X^*, Y^*, X, Y are fixed and variable coordinates.

In [2] on the basis of analyzing experimental data and known solutions of problems of heat conduction and the electromagnetic field in the interelectrode gap, it was shown that Eq. (1) is not always satisfied to a given accuracy, but only for a strictly defined set of thermal, geometric, and other parameters.

Establishing the necessary and sufficient conditions for satisfying Eq. (1) is stimulated by the solution of a series of applied problems, especially for those experimental cases when it is complicated to measure, for example, the temperature inside an active element.

THEOREM. Let a function of internal heat sources have the form

$$W(X, Y) = W_1(X) W_2(Y), \tag{2}$$

where $W_1(X)$ and $W_2(Y)$ are continuous functions, which satisfy the differential equations

$$W_{1XX} = -\mu_i^2 W_1, \quad W_{2YY} = -\mu_i^2 W_2, \quad 0 < X < 1, \quad 0 < Y < R, \tag{3}$$

for the boundary conditions

$$\begin{gathered} W_X(0, Y) - \mathrm{Bi}_2 W(0, Y) = 0; \quad W_X(1, Y) + \mathrm{Bi}_1 W(1, Y) = 0; \\ W_Y(X, 0) - \mathrm{Bi}_4 W(X, 0) = 0; \quad W_Y(X, R) + \mathrm{Bi}_3 W(X, R) = 0, \end{gathered} \tag{4}$$

and μ_i are eigenvalues. They are the roots of two transcendental equations

$$\mathrm{ctg}\,\mu_i = \frac{\mu_i^2 - \mathrm{Bi}_1\mathrm{Bi}_2}{\mu_i(\mathrm{Bi}_1 + \mathrm{Bi}_2)}, \tag{5}$$

$$\mathrm{ctg}\,\mu_i R = \frac{\mu_i^2 - \mathrm{Bi}_3\mathrm{Bi}_4}{\mu_i(\mathrm{Bi}_3 + \mathrm{Bi}_4)}. \tag{6}$$

Then the equation of the temperature field in the fuel element will have the form

$$\Theta(X, R) = \frac{1}{2\mu_i^2} \mathrm{Po}_0 W_1(X) W_2(Y). \tag{7}$$

Here this function satisfies Eq. (1), Poisson's equation

$$\Theta_{XX} + \Theta_{YY} = -\mathrm{Po}_0 W_1(X) W_2(Y) \tag{8}$$

and boundary conditions that are analogous to Eq. (4).

Proof. The solution of Poisson's equation (8) for the boundary conditions of a solid by the method of finite integral transforms [3] has the form

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TABLE 1. Discrete Series of Values of R for a Pressure Plate of a Turbo Generator for Which Eq. (1) is (or is not) Fulfilled with an Error $\varepsilon \le 5\%$. The Initial Data from [2, 4]: $Bi_1 = 0.8$, $Bi_2 = 1.6$, $Bi_3 = 0.4$, $Bi_4 = 1.2$; Calculation according to Eq. (5): $\mu_1 = 1.3827059$ ($\Delta_1 = 1.4 \cdot 10^{-6}$)

R	Δ_2, 10^{-5}	(R)	Δ_2
0,7206	—0,53	2,2720	—11793
2,9927	—8,1	2,2725	+1648
5,2647	+3,9	4,52	—30,6
7,5368	—3,7	9,0	—8,8
9,8088	+8,3	9,1	+60,8

TABLE 2. Recommended to Examination Relative to Eq. (1) Range of Values of R for Magnetic Conductor of a Charged Particle Accelerator [2]: $Bi_{1,2} = 8.62$, $Bi_{3,4} = 1.38$, $Po_0 = 19.07$, $\mu_1 = 2.563477$ ($\Delta_1 = -9 \cdot 10^{-7}$)

R	Δ_2	R	Δ_2	R	Δ_2
0,35	0,14	2,75	0,38	5,25	0,15
⋮		⋮		⋮	
0,48	—0,30	3,0	—0,49	5,5	—0,62
1,55	0,25	3,95	0,52	6,5	0,10
⋮		⋮		⋮	
1,7	—0,3	4,20	—0,43	6,6	—0,30

$$\Theta(X,\ Y) = \sum_{n=1}^{\infty} \frac{\tilde{\Theta}(\mu_n,\ Y)\, K(\mu_n,\ X)}{\int_0^1 K^2(\mu_n,\ X)\, dX}, \tag{9}$$

where $K(\mu_n, X) = \mu_n \cos \mu_n X + Bi_2 \sin \mu_n X$ is the kernel of the finite integral transform with respect to the coordinate X, μ_n are eigenvalues which are determined from Eq. (5), and

$$\begin{gathered}
\tilde{\Theta}(\mu_n,\ Y) = \mathrm{Po}_0 F(\mu_n)\, L(\mu_n,\ Y)/\mu_n^2, \\
F(\mu_n) = \int_0^1 W_1(X)\, K(\mu_n,\ X)\, dX, \\
L(\mu_n,\ Y) = \Psi(\mu_n,\ Y) + C_1 n_1(Y) + C_2 n_2(Y), \\
\Psi(\mu_n,\ Y) = W_2(Y) + \varphi(\mu_n,\ Y), \\
\varphi(\mu_n,\ Y) = -\exp(-\mu_n,\ Y) \int \xi(Y) \exp(2\mu_n Y)\, dY, \\
\xi(Y) = \int \frac{d^2 W_2(Y)}{dY^2} \exp(-\mu_n Y)\, dY, \\
C_1 = \frac{1}{d}\left[\frac{d\Psi(0)}{dY} - \mathrm{Bi}_4 \Psi(0)\right], \quad C_2 = -\frac{1}{d}\left[\frac{d\Psi(R)}{dY} + \mathrm{Bi}_3 \Psi(R)\right], \\
n_1(Y) = (\mu_n + \mathrm{Bi}_3)\exp(-\mu_n Y) - (\mu_n - \mathrm{Bi}_3)\exp[-\mu_n(2R - Y)], \\
n_2(Y) = (\mu_n + \mathrm{Bi}_4)\exp[-\mu_n(R - Y)] + (\mu_n - \mathrm{Bi}_4)\exp[-\mu_n(R + Y)], \\
m_n = \exp(-2\mu_n R), \quad d = \mu_n(\mathrm{Bi}_3 + \mathrm{Bi}_4)(m_n + 1) + (\mu_n^2 + \mathrm{Bi}_3\mathrm{Bi}_4)(m_n - 1).
\end{gathered} \tag{10}$$

Solution (9) taking into account all of the expressions in Eqs. (10) is valid if the functions $W_1(X)$ and $W_2(Y)$ are continuous. From this class of functions it is possible to find those harmonic functions which satisfy the conditions of the theorem:

$$W_1(X) = \mu_i \cos \mu_i X + \mathrm{Bi}_2 \sin \mu_i X, \tag{11}$$

TABLE 3. Range of Values (R) Not Recommended for Practical Applications. Initial Data Are Cited in Table 2

(R)	Δ_2	(R)	Δ_2	(R)	Δ_2
1,2	—16	3,5	—2,7	6,0	3,6
1,3	4,5	3,7	16	6,3	1,5
2,4	—8,3	4,7	—2,4	6,9	—1,1
2,6	1,8	5,0	3,2	7,5	1,9

TABLE 4. Distribution of $\varepsilon(X, Y)$, % for a Pressure Plate of a Turbo Generator. Initial Data Are from [2, 4]: $Bi_1 = 0.8$, $Bi_2 = 1.6$, $Bi_3 = 0.4$, $Bi_4 = 1.2$; $R = 7.5$, $Po(X, Y) = Po_0 \exp(-NX)(1 + MY + DY^2)$; $Po_0 = 112$, $N = M = 0.0$, $D = -1.778 \cdot 10^{-2}$ (calculation per Eq. (16)).

X	Y				
	0,0	0,25R	0,5R	0,75R	R
0,0	2,56	1,75	1,64	1,64	—2,64
0,25	1,25	0,96	0,90	0,87	—1,19
0,5	—0,24	0,04	0,04	0,07	0,23
0,75	—0,58	—1,27	—0,11	—0,11	0,59
1,0	—2,71	—1,77	—1,68	—1,61	2,56

$$W_2(Y) = \mu_i \cos \mu_i Y + \mathrm{Bi}_4 \sin \mu_i Y. \tag{12}$$

Then after substituting functions (11) and (12) into solution (9) and recalling Eqs. (3)-(6) and all of Eqs. (10) we obtain

$$\varphi(Y) = -\frac{1}{2} W_2(Y), \quad \Psi(\mu_n, Y) = \frac{1}{2} W_2(Y), \; C_1 = 0, \; C_2 = 0.$$

On the basis of the orthogonal properties of the eigenfunctions (9), (11), (12) we find

$$\Theta(X, Y) = \frac{1}{2\mu_i^2} \mathrm{Po}_0 W_1(X) W_2(Y),$$

which coincides with function (7).

We shall now dwell on the proof of the following proposition: if the dimensionless temperature exactly satisfies Eq. (1), then it is found from the solution of Poisson's equation (8) and boundary conditions analogous to Eq. (4).

As is evident from Eq. (1), the desired function is the product of two functions, each of which depends on one variable. Let there be some function

$$\Theta(X, Y) = \eta(X) G(Y), \tag{13}$$

which, on substitution into Poisson's equation (8), leads to the following form:

$$\frac{\eta_{XX}}{\eta} + \frac{G_{YY}}{G} = -2\mu^2, \tag{14}$$

where μ^2 is the separation parameter of the variables.

It is obvious that each term on the left-hand-side of Eq. (14) is also a constant, i.e.,

$$\frac{\eta_{XX}}{\eta} = -\mu^2, \; \frac{G_{YY}}{G} = -\mu^2. \tag{15}$$

The solution of two ordinary second-order differential equations for third-order boundary conditions of the type of Eq. (4) have the form

$$\eta(X) = \mu_i \cos \mu_i X + \mathrm{Bi}_2 \sin \mu_1 X, \; \Psi(Y) = \mu_i \cos \mu_i Y + \mathrm{Bi}_4 \sin \mu_i Y.$$

Here μ_i are eigenvalues which are found from Eqs. (5) and (6).

On the other hand, according to Eq. (8), (13), and (14):

$$\Theta(X, Y) = \frac{1}{2\mu_i^2} \mathrm{Po}_0 W_1(X) W_2(Y),$$

but this is only possible when the function for the heat sources satisfies differential equation (3) and boundary conditions (4).

We now turn our attention to satisfying condition (6). As is known, the solution of transcendental equation (5) has a set of roots $\mu_1 < \mu_2 < \mu_3 < \ldots < \mu_i$. Therefore each given value of R will correspond to definite values of μ_i. In the opposite case, Eq. (1) is not satisfied.

Thus, for exact fulfillment of the coupling relation between dimensionless (excess) temperatures in the fuel element (1), it is necessary and sufficient to satisfy conditions (2)-(8) and the theorem is proven.

Discussion of the Results. In real cases $W_1(X)$ and $W_2(Y)$, as a rule, are rarely subject to the conditions of the theorem. This is because the nature of the origin of heat in fuel elements can be different: the liberation of Joule heat with the passage of electric current in normal conductors, heat losses from hysteresis and vortex currents in the magnetic circuits in electric machines, heat liberated as a result of nuclear reactions in atomic reactors, etc.

Therefore, it should be expected that Eq. (1) should be satisfied with some uncertainty

$$\varepsilon(X, Y) = \frac{\Theta(X, Y) - \Phi(X, Y)}{\Theta(X, Y)} \cdot 100\% \leqq \varepsilon_{\text{preset}}, \quad (16)$$

where $\Theta(X, Y)$ is the dimensionless temperature, found from experiment or calculated from an exact solution; $\Phi(X, Y)$ is suitable for Eq. (1) or an approximate solution, if, for example, Eq. (9) is limited to one first term of the series.

In practice, only fuel element variants are of interest when the maximum value of $\varepsilon(X, Y) \leq \varepsilon_{\text{preset}}$. They can be quickly verified according to [2] by the existence of relation (1) from experimental excess surface temperatures of the fuel elements. However, this is related with the "sorting" of different fuel elements, i.e., for one of them the error will be $\varepsilon < \varepsilon_{\text{preset}}$, and for the others $\varepsilon > \varepsilon_{\text{preset}}$. Therefore, a problem arose in determining the range of geometric dimensions of the fuel element of rectangular cross-section, for which the minimum error in reproducing the temperature field is expected according to Eq. (1). We shall denote the discrepancy between the left- and right-hand-sides of Eqs. (5), and (6) through Δ_1 and Δ_2, respectively. Numerical calculations (Table 1) established that for each μ_i there exists a discrete series of values of R, $R_1 < R_2 < R_3 < \ldots R_i$, for which $\Delta_1, \Delta_2 \to 0$. This enables one to choose in advance predetermined dimensions at the selection stage of a projected variant of a fuel element (Tables 2, 3) with subsequent experiment or numerical verifications of Eq. (1), which opens the possibility of obtaining reliable information about the thermal state, for example, of a pressure plate of a turbo generator [4] (Table 4).

NOTATION

X, Y are dimensionless coordinates, Po_0 is the Pomerantsev number, Bi_j (j = 1, 2, 3, 4) is the Biot number.

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